Universal Journal of Educational Research 7(12): 2584-2596, 2019 http://www.hrpub.org
DOI: 10.13189/ujer.2019.071205

Challenges and Prospective Directions of Enhancing Teaching Mathematics Theorems in School

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Received August 20, 2019; Revised September 29, 2019; Accepted November 8, 2019

Abstract Students master the methods of reasoning and argumentation in school mathematics when study theorems and their proofs. In studying theorems, students must learn to understand the essence of the theorem. For this, special work with the wording of the theorem is necessary. We believe that this work must contain: a) four mandatory stages; b) three stages for students who take a genuine interest in mathematics; c) stage for those who want to go further in learning mathematics. The aim of this study is to investigate the challenges in teacher practices during four mandatory stages; to reveal the obvious and hidden factors that affect student outcomes. For this study, a survey that involved 129 mathematics teachers across Ukraine was carried out. A standard closed ended questionnaire was developed. Several challenges were found in the area of students' motivation, and teachers' lack of understanding of how to most efficiently teach the wording of theorems. Factor analysis was used to analyze the results of the survey, and 10 variables were found to describe how teachers motivated students to learn theorems and taught the wording of theorems.

Keywords Teaching Mathematics, Teaching a Theorem, Teaching and Learning the Wording of a Theorem

1. Introduction

One of the high priority goals in school instruction is the development of students' logical reasoning that allows them to provide non-contradictory, consistent, and evidence-based reasoning. The methods of consistent logical reasoning and argumentation and their components (separate actions and operations) are formed, in particular, when students prove mathematical statements. Nowadays, survey results claim that secondary education applicants seem to lack the skills necessary to articulate clear arguments when providing mathematical proofs. These skills seem to be completely foreign to school students. Students explain such neglect by the fact that the tasks to prove a mathematical statement are rarely included in the final external state assessment in mathematics.

Indicative are the research findings obtained during the All-Ukrainian monitoring survey administered to principals and secondary school teachers (based on the TALIS methodology) (Shchudlo, Zabolotna, & Lisova [1]). The study summarized mathematics teachers' beliefs about the necessity of teaching how to prove the mathematical facts. The obtained data reveal that Ukrainian mathematics teachers generally consider the student ability to reason logically to be a significant educational outcome. While 72.6 % of the Ukrainian mathematics teachers believe that logical thinking and argumentation skills are more important than the content of the discipline, this indicator is much higher for teachers from TALIS countries and stands at 83.5 %.

Moreover, the survey results illustrate a negative trend: a much lower percent of younger teachers believe that student's ability to think and reason in mathematics should be an important outcome in teaching mathematics. Among teachers aged over 60, such beliefs are supported by 82.1 % of respondents interviewed, the number of teachers aged 50 and 59 accounted for 73.8 %, the percentage of 40-49 year old teachers was insignificantly higher (74.9 %). Teachers aged 30 to 39 made up 67.1%, and the number of those who were under 29 years old accounted for 64.9 %. This trend is of a particular concern, because newly qualified teachers are more able to influence their students. Consequently, students will also neglect the importance of reasoning in learning mathematics.

1.1. Background of the Study

It is a common fact that theorems in the school course of mathematics are formulated in the form of statements. The term ‘statement’ as a logical-semantic category affirms or denies the expression of language (form of reasoning). These declarations state the class of empirical or abstract objects, detect the relationship between objects of thought, identify the presence or absence of properties in a class of objects or elements of a certain class. It should be possible to say whether this declaration is true or false.

General methodological aspects of the teaching of proofs of the theorems were considered by Bradys, Minkowski, & Kharcheva [8], Grudionov [9], Polya [12], Metelsky [10], Ball, Hoyles, Jahnke, & Movshovitz-Hadar [14], Slepkan [2], Tarasenkova [11], Dalinger [3], Franklin & Daoud [13], Hanna [31], and others. The problem of teaching the proofs of mathematical statements was studied by scientists in the following lines of investigation: the psychological and pedagogical bases of teaching students proofs (Slepkan [2]; Stylianides [28]; Boero, Garuti, Lemut, & Mariotti [25]; Yackel & Hanna [29]; and others), the methods of designing and teaching proofs in elementary classes (Ball & Bass [15; 16]; and others), in the school algebra (Healy & Hoyles [17]), the methods of designing and teaching proofs in the school geometry (Tarasenkova [11], Reiss, Hellmich, & Reiss [27]), the development of senior pupils’ skills to prove mathematical statements in the process of learning algebra and the principles of analysis (Kugai [6]), the application of heuristics in search of the mathematical proof method (Scafa [18]), the formation and development of logical skills in teaching students math in depth (Tarasenkova & Akulenko [19]), the formation of students' skills of proving mathematical statements when learning functions in-depth (Kirman [21]), the teaching of the elements of mathematical logic and the theoretical foundations of the math statements proofs (Akulenko & Leshchenko [22]), teaching proofs in the in-depth learning of stereometry (Yatsenko [20]), and others.

Modern researchers focus their attention on the benefits and warnings regarding the use of computers in teaching and learning proofs, based on computer experiments in particular (Shirikova [23]), analyze teaching various types of reasoning and methods of proof related to ideas about constructing knowledge in mathematics as a discipline (Ball & Bass, 2003 [16]), describe mathematical reasoning components that include communication, basic skills, connection, and logical thinking [34], elaborate how pedagogy can be designed to develop students’ understanding of the proof (e.g. Rowland [32]; Hanna [31]; Stylianides & Stylianides [30]; Knuth [26]; and others), research comprehensive perspectives on the learning and teaching of proof (Harel & Sowder [33]), study the dependence of cognitive unity on the specific rationality (e.g. analytic geometry rationality, or synthetic geometry rationality) according to which a conjecturing and proving problem is dealt with (Boero [24]). Despite the wide range of pedagogical, psychological, and methodological studies, the problem of teaching the methods of proof and the methods of searching a proof of mathematical statements remains relevant in school practice.

The methods widely used in theoretical investigations and in practice for teaching students mathematical facts and their proofs include:

1) analysis and study of the proofs, demonstrated by the teacher or described in the textbook, followed by their subsequent reproduction; student independent finding of proof solutions using analogy with the given proofs; student independent finding of proof solutions using demonstrations; independent search and conducting of proofs (Slepkan [2]; Dalinger [3]; Weber & Alcock [4]; and others);
2) analysis of the solved proof and its presentation; unassisted discovery of facts, finding the solution of the individual proof; the refutation of the proposed proof (Sarantsev [5]; Kugai [6]; and others);
3) analysis and study of shared proofs; identifying the logical fundamentals of proofs and presenting them to students; student unassisted finding of proof solutions using analogy or with the teacher’s help; unassisted finding of proof solutions using the knowledge of the proof logic (Stolyar [7]).

When applying these practices to teach students theorems and their proofs, present working teachers traditionally follow the sequence:

1) motivating to learn a theorem;
2) presenting a theorem;
3) working with the wording of a theorem;
4) motivating to prove a theorem;
5) searching with students the way to prove a theorem;
6) presenting reasoning and proving;
7) looking over the proof;
8) applying the learned proof solutions to prove other mathematical statements.

Our practical experience proves that the difficulties faced by students, their reluctance and the inability to search and provide the proof of the theorems are caused as a response to certain teaching strategies. Teachers tend to neglect motivating students to learn the methods of proving a theorem. Motivating for learning and motivating for proving are the two fundamentally different stages; each has its own specific goals, features that characterize its implementation and require the corresponding instructional support. In addition, the survey indicates that students often do not understand the content and mathematical essence of the theorem to be proved. This is caused by the teachers' insufficient awareness about the significance of teaching the wording of a theorem. It is just at this stage when students find out what is given in the theorem, what it is necessary to prove, what data are presented explicitly in the wording of the theorem and what data are hidden.

Here is an example. The theorems in the school mathematics course can be formulated in a categorical or implicational form. The categorical form tends to sound like the following: "The sum of the two adjacent angles is equal 180°", or "Vertical angles are equal". The same theorems in the implicational form sound like this: "If the angles are adjacent then the sum of their degree measures is 180°", "If the angles are vertical, then they are equal".

The implicational stated theorems are formalized in the following way: $(\forall x \in M)(S(x) \Rightarrow P(x))$, where $S(x)$ is the hypothesis of the theorem, $P(x)$ is the conclusion of the theorem. The list of elements of the set M with predicates $S(x)$ and $P(x)$ can be found in the explanatory part of the theorem.

The explanatory part of the theorem answers the question: "What do we consider in the theorem?". The hypothesis asks "What is given?". The conclusion requires "What should I prove?". The explanatory part of the theorem is often presented implicitly. In the following example of the theorem which says: "Through a point outside of this straight line in space a straight line, parallel to this line, and only one line can be drawn", the explanatory part is worded in the following way: "We consider assertions on the set of pairs of points and lines in space (straight line a, point A)". The hypothesis reads "Given point A does not lie on given line a". The conclusion says: "There exists straight line b that is passing through given point A running parallel to given line a, and this line is the only one".

If the wording of the theorem contains one hypothesis and one conclusion, then the theorem is called simple, for example: "The sum of the two adjacent angles is equal 180°". If the wording of the theorem contains several hypothesis or several conclusions, this theorem is called composite, for example: "Two straight lines that run parallel to the third one, are parallel with each other", "If p is a prime number, then an arbitrary natural number a is divided by p, or a and p are relatively prime natural numbers".

In the composite theorems, the components of the hypothesis or conclusions can be related by means of the conjunctions (logical conjunction "and") or disjunctions (logical conjunction "or"). These connectors can be represented explicitly or implicitly in the wording of the theorem.

For example, here is an example of the theorem: "Two planes that run parallel to the third one are parallel with each other". Its explanatory part can be worded like this: "We consider allegations on the set of triple planes (α, β, γ) in space". The hypothesis is: "Plane α runs parallel to plane γ and plane β runs parallel to plane γ". The conclusion says: "The plane α runs parallel to plane γ". The theorem is composed, since the hypothesis consists of two parts that are connected by a conjunction, however, this connection in the wording of the theorem is represented implicitly.

Here is another example. The theorem: "If p is a prime number, then an arbitrary natural number a is divided by p, or a and p are relatively prime". Its explanatory part is: "The statement is considered on the set of pairs of natural numbers (a and p are natural numbers)". The hypothesis says: "Number a is an arbitrary positive integer, and number p is an arbitrary prime number". The conclusion states: "Number a is divided by p, or a and p – are relatively prime". The theorem is composed, since both the hypothesis and the conclusion contain two components. The components in the hypothesis are connected by conjunction; the components in the conclusion are connected by disjunction and explicitly presented in the wording of the theorem.

We distinguish 8 stages of the teaching and learning the wording of the theorem:

1) establishing the form of the statement (categorical, implicational);
2) singling out the explanatory part, hypothesis, and conclusion;
3) creating a short record of the theorem;
4) re-wording of the theorem "in own manner";
5) formulating the converse of the theorem statement, the opposite statement, the converse of the opposite statement and, if possible, verifying its truth;
6) choosing an equivalent statement (if there is any) among several proposed statements;
7) formulating the statement equivalent to the theorem, if possible;
8) some accessorial work at the wording of the theorem, like: wording the theorem by putting the scrambled words into the correct order, finding and correcting the mistakes in the proposed wording of the theorem etc.

Stages 1-4 are mandatory. They provide for training the work patterns which should be mastered by every student in the group. These stages are a part of the lesson at which the theorem is introduced and take only 5 minutes of the lesson, en masse. Stages 4-7 are aimed at those students who take a genuine interest in mathematics and want to improve their knowledge. These stages are implemented through individual assignments for classroom and home work. Stage 8 is actualized at the following lesson.

When students are working with the wording of the theorem they find out the gist of the theorem, since it highlights the set of objects the theorem is projected on, what is known about these objects, and what property or feature has to be proved.

1.2. Problem of the Study

Since the research findings reveal that the formation of students' ability to form consistent, evidence-based reasoning tends to be neglected, there is a need to rethink the traditional methods of teaching students mathematical facts and their proofs. Being multifaceted, this problem can be solved if the 'point of departure' coordinates (which identify the problem using certain parameters), the 'point of application' (strategic direction) and 'action vector' (a set of didactically reasonable educational innovative support to teach students the theorems and their proofs) are identified.

1.3. The Objective of This Study is

1) to investigate what and how traditional practices are used to teach students to prove mathematical facts (especially in the first three stages);
2) to identify the obvious and hidden factors that affect the productivity at these stages when teaching students the mathematical facts and their proofs;
3) to elaborate and present some prospective directions for improving the teaching practice in motivating students to learn the theorem and it's proof and for organizing the work with the wording of the theorem.

2. Materials and Methods

2.1. Study Design

This study is a descriptive cross-sectional research carried out at the Educational-Scientific Institute of Information and Educational Technologies at Bohdan Khmelnytsky National University in Cherkasy and the Faculty of Physics and Mathematics at Kryvy Rih State Pedagogical University.

The study area. The research was conducted in Ukraine. The study population: It includes mathematics teachers across Ukraine.

Sampling. Total coverage sample includes 129 teachers. The conclusion about the representativeness of the sample is made on the basis of the analysis of the range of the oscillations of the answers to individual questions of the questionnaire. We took such a basis for the conclusion; since sampling was random, the scale of measurement of the answers to the questions was either nominal or ranked. Under these conditions, the distribution of the response range of the received sample response reflects the distribution of the response oscillations in the aggregate [35]. The overall response rates of the questionnaire differ by ± 5% from those obtained in our survey. The teachers were distributed in the following way: 52.8% of teachers teach K 7; 56% of teachers teach K 8; 53.6% teach K 9; 37.6% teach K 10; 39.2% teach K 11. The term of work at school for teachers varies. 8.8% of respondents have up to 5 years of experience; 13.6% teachers have from 5 to 10 years of work experience; 26.4% teachers' experience ranges from 10 to 20 years; and 51.2% of teachers have worked for more than 20 years. Thus, the survey was administered to efficient, experienced teachers who have an established personal teaching philosophy and have acquired substantial experience in teaching students of theorems and their proofs.

2.2. Data Collection

Data collection tools: A standard closed ended questionnaire was developed by the researches based on available literature to evaluate the teachers' beliefs and attitudes toward teaching students of theorems and their proofs.

Data collection technique: The data was collected within one month. Every participant filled the questionnaire by him / herself. Each questionnaire took from 10 to 15 minutes to be filled out, there were no missing questionnaires.

Ethical considerations: The purpose of the study was explained to every participant and the participants were reminded that the information should be confidential and used only for the purpose of the study.

Data analysis technique: The data was analyzed manually by simple statistical method and presented in the forms of tables, graphs and figures.

Factor analysis was used to process the survey results. It helped:

1) to examine the relationships of the input variables (each grouping of variables is determined by the factor that gives them the maximum);
2) to identify factors that cause the relationship of input variables;
3) to calculate the numerical values of factors as new, integral variables.

Factor analysis followed the sequence:

1) the correlation matrix for all variables was calculated (in our case, data received from the teachers who participated in the analysis);
2) factors were separated by using the main components analysis method;
3) factors were rotated to simplify the structure (Varimax rotation with Kaiser normalization was used);
4) factors were interpreted; the SPSS 19.0 software package was used.

3. Results

The survey was aimed at identifying the emphasis in teacher's beliefs about the teaching and learning the theorems and their proofs in the school geometry (K 7 – K 9). Almost all teachers are convinced that there is a need to teach theorems in the school geometry. Speaking about the necessity of teaching and learning of the proofs, the opinion of the polled teachers is not so unambiguous. 64.8% of the polled teachers believe that the greatest contribution of the proofs' teaching and learning can be seen into the development of student cognitive abilities, while 60.8% of respondents think that this practice develops student logical thinking. It is noted that 42.4% of teachers are convinced that the proving the theorems demonstrates the structure of the deductive method that is used to construct a mathematical theory. 28.8% of respondents believe that learning proofs can create conditions for the use of heuristics in the teaching and learning mathematics; and 24% of respondents emphasize the ideological value that proofs have in developing students' general cultural competence. The positive influence highlighted by the survey participants is the fact that by learning how to prove mathematical statements students also learn how to think logically.

On the other hand, it is rather disturbing that only 24% of respondents focus their attention on the methodological, ideological, and cultural significance of the theorem proofs. In our opinion, this can lead to a complete refusal to study the proofs of the theorems that is exhibited by students enrolled in social science and liberal art classes (K 10 – K 11). This can be explained by the catastrophic lack of the instruction time allocated to teach mathematics. Unfortunately, in Ukraine, instruction time for algebra and geometry in social science and liberal art classes is completely insufficient namely three 45 minute lessons per week according to the state standards.

Since teaching theorem proofs should begin with motivating students to study, followed by teaching the wording of the theorem, teachers were asked a series of relevant questions. The survey revealed that 65.1% of respondents regularly motivated students to study theorems in the school geometry, 35.9% of respondents did this irregularly. Therefore, the inadequate motivation to involve students in studying theorems is becoming the major obstacle for the students to acquire mathematical and cognitive skills. The techniques used by teachers to motivate the study of theorems are marked as variables. The most preferred technique that teachers use to motivate students to study theorems is to emphasize the practical value of this theorem. 76.7% of teachers realize the importance of highlighting the practical application of the theorem for solving everyday problems of an applied nature (use it); 41.1 % of teachers accentuate on the practical significance of the theorem to prove other theorems. Teachers appear to turn to historical facts associated with the name of scientists, in whose honor the theorem is called, quite infrequently (32.6%). Motivating students to study theorems by constructing and studying them is preferred by 28.7% of the teachers surveyed, while 19.4% of teachers tend to create in class conditions for the natural phenomena simulation and generalization of observations in order to motivate students to study theorems.

Working with the wording of the theorem is aimed at making students understand the essence and the idea of the mathematical fact established by the theorem of. 71.3% of the teachers always allocate time for this; 27.9% of teachers do this kind of work episodically; and 0.8% of respondents do not do anything related to working with the wording of the theorem. When working with the wording of the theorem, teachers use the following techniques:

1) establish the form of the formulation (categorical, implicational) together with students (27.9%);
2) distinguish the explanatory part, hypothesis and conclusion (56.6%);
3) distinguish the hypothesis and conclusion (39.5%);
4) distinguish only the hypothesis of the theorem (3.9%);
5) formulate the converse of a theorem and, if possible, verify its truth together with students (14%);
6) use the opposite statement (10.1%);
7) use the converse of the opposite to a theorem (15.5%);
8) ask students to choose one from among the proposed statements that is equivalent to the theorem (31.8%);
9) formulate with students the statement which is equivalent to the theorem (15.5%);
10) offer students to form the wording of the theorem from the given words (33.3%);
11) offer to students to find and to correct mistakes in the proposed wording of the theorem (27.1%).

Teachers in their responses noted that students have difficulties in providing these techniques, including:

1) 43.4% of teachers stated strongly that students can't turn the categorical statement into implicative;
2) 35.7% respondents witness that students failure to formulate the converse of a theorem;
3) 27.1% of teachers pointed out that students can't formulate the opposite statement;
4) 29.5% respondents revealed that students were unable to find and separate the explanatory part and the hypothesis, though they can allocate the conclusion;
5) 25.6% of teachers stated that students can't put the conclusion in words, though they can allocate the explanatory part and the hypothesis;
6) 14.7% of teachers stated that students can't allocate either the hypothesis or the conclusion.

The empirical data obtained required more consideration. In order to reveal the structure of the interrelations between the variables obtained during the survey, we calculated the correlations (r-Spirmen) for each pair of variables and presented them as the correlation matrix (Table 1).

Table 1. Correlation matrix

	V_1	V_2	V_3	V_4	V_5	V_6	V_7	V_8	V_9	V_{10}
V_1	1.000	0.463**	0.805**	0.546**	0.491**	0.585**	-0.167	0.600**	-0.213*	0.175*
V_2	0.463**	1.000	0.469**	0.524**	0.608**	0.695**	-0.511**	0.745**	-0.375**	-0.293**
V_3	0.805**	0.469**	1.000	0.655**	0.600**	0.671**	-0.188*	0.611**	-0.321**	0.208*
V_4	0.546**	0.524**	0.655**	1.000	0.852**	0.832**	-0.306**	0.516**	-0.220*	0.150
V_5	0.491**	0.608**	0.600**	0.852**	1.000	0.853**	-0.350**	0.566**	-0.287**	0.112
V_6	0.585**	0.695**	0.671**	0.832**	0.853**	1.000	-0.421**	0.634**	-0.241**	0.092
V_7	-0.167	-0.511**	-0.188*	-0.306**	-0.350**	-0.421**	1.000	-0.339**	0.123	0.278**
V_8	0.600**	0.745**	0.611**	0.516**	0.566**	0.634**	-0.339**	1.000	-0.619**	-0.091
V_9	-0.213*	-0.375**	-0.321**	-0.220*	-0.287**	-0.241**	0.123	-0.619**	1.000	0.136
V_{10}	0.175*	-0.293**	0.208*	0.150	0.112	0.092	0.278**	-0.091	0.136	1.000

Note:

** the correlation is significant at the level of 0.01 (2-sided);

* the correlation is significant at the level of 0.05 (2-sided);

V_1 – a teacher motivates students to study a theorem;

V_2 – a variety of techniques used by the teacher to motivate students to study theorems;

V_3 – allocation of additional instruction time to practice the wording of the theorem;

V_4 – to identify the explanatory part, hypothesis and conclusion of the theorem;

V_5 – to define the form of the theorem statement (categorical, implicational);

V_6 – to formulate the converse of a theorem;

V_7 – to formulate the opposite statement;

V_8 – to formulate the converse of a opposite statement;

V_9 – a variety of techniques used by the teacher in teaching the wording of the theorem;

V_{10} – the number of difficulties students experience when working with the wording of the theorem.

Graphical representation of the structure that illustrates statistically significant connections is presented in the form of a correlation galaxy (Figure 1).

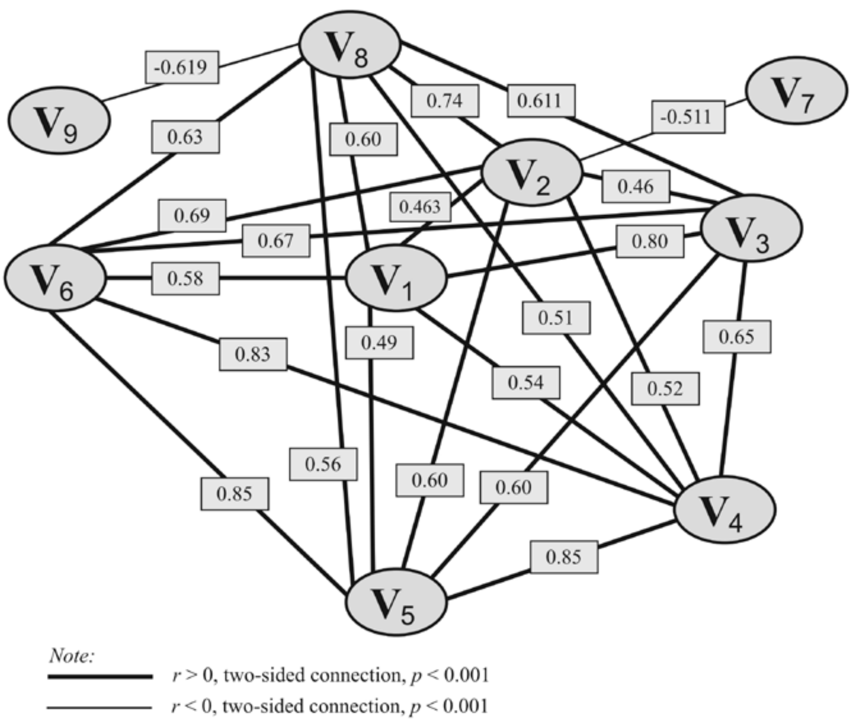

Figure 1. Correlation galaxy

To determine the efficiency of the time spent on working with the wording of the theorem, the dependence between variables such as the time that the teacher allocates to this work (V_3), and the number of difficulties that students experience when working with the wording of the theorem (V_{10}) was studied. The results are demonstrated in the table of crosstabulation (Table 2).

Table 2. Crosstabulation

		**Indicate the number of students' difficulties when working with the wording of the theorem			Total
		1 difficulty	2 difficulties	3 and more difficulties	
*Do you take time to teach student to formulate a theorem?	Frequency	17_a	73_b	$2_{a, b}$	92
	Yes, always (%)	18.5%	79.3%	2.2%	100.0%
	Frequency	16_a	20_b	$0_{a, b}$	36
	Yes, sometimes (%)	44.4%	55.6%	0.0%	100.0%
	Frequency	1_a	0_a	0_a	1
	No (%)	100.0%	0.0%	0.0%	100.0%
Total	Frequency	34	93	2	129
	Total (%)	26.4%	72.1%	1.6%	100.0%

Taking into consideration the data received, Pearson's criterion χ^2 was used (Table 3).

Table 3. (χ^2 Chi-Square Tests)

Criteria	Value	df.	Asymp. Sig. (2-sided)	Exact Sig. (2-sided)	Exact Sig. (1-sided)	Probability at the point
Pearson Chi-Square	12.256[a]	4	0.016	0.018		
Likelihood Ratio	12.242	4	0.016	0.006		
Fisher's exact criterion	13.498			0.006		
Linear-by-Linear Association	11.640[b]	1	0.001	0.001	0.001	0.001
N of Valid Cases	129					

The obtained data (Table 4) indicate that there is a certain relationship between variables: the time that the teacher allocates to teach the wording of the theorem and the number of difficulties faced by students when working with the wording of the theorem. However, the value of the correlation coefficient (– 0.293) suggests that there is a very weak negative correlation between these variables; that is, the more time is spent by the teacher practice the wording of the theorem, does not always leads to the fewer difficulties for the students.

Table 4. Symmetric Measures

		Value	Asympt. Std. Error [a]	T[b] Approx. T	Approx. Sig.	Value
Ordinal by Ordinal	Kendall's tau-c	– 0.179	0.058	– 3.102	0.002	0.001
	Gamma	– 0.594	0.135	– 3.102	0.002	0.003
	Spearman Correlation	– 0.293	0.089	– 3.458	0.001[c]	0.001
N of Valid Cases		129				

It is also of scientific interest to establish the relationship between the variety of techniques used by the teacher when working with the wording of the theorem (V_9), and the number of difficulties students experience doing this task (V_{10}) (Table 5).

Table 5. Crosstabulation

		*Indicate the number of difficulties that students experience when working with the wording of the theorem			Total
		1 difficulty	2 difficulties	3 and more difficulties	
**Indicate what techniques do you use when working with the wording of the theorem	Frequency	2_a	0_a	0_a	2
	1 technique (%)	100.0%	0.0%	0.0%	100.0%
	Frequency	16_a	36_a	1_a	53
	2 techniques (%)	30.2%	67.9%	1.9%	100.0%
	Frequency	16_a	57_a	1_a	74
	Three and more techniques (%)	21.6%	77.0%	1.4%	100.0%
Total	Frequency	34	93	2	129
	Total (%)	26.4%	72.1%	1.6%	100.0%

Table 6. (Chi-Square χ^2 Tests)

	Criteria	Value	df.	Asymp. Sig. (2-sided)	Exact Sig. (2-sided)	Exact Sig. (1-sided)
Pearson Chi-Square	6.948^a	4	0.139	0.100		
Likelihood Ratio	6.717	4	0.152	0.131		
Linear-by-Linear Association	3.009^b	1	0.083	0.107	0.060	0.032
N of Valid Cases	129					

It can be seen from Table 5 that all data show the insignificant statistical difference ($p > 0.05$). This is also indicated by χ^2 Pearson Tests (6.948) at $p > 0.05$ (Table 6).

Thus, it is concluded that there is no statistically significant relationship between the variety of techniques used by the teacher when working with the wording of the theorem (V_9) and the number of difficulties students experience in this work (V_{10}). Therefore, the teachers' passion to use a variety of different methods at this stage of teaching of theorem is not a valid factor for eliminating the difficulties students experience doing this task.

Factor Analysis. Factor Analysis was used to identify the explicit and hidden factors that influence the effectiveness of the teacher's techniques at the initial stages of teaching the theorems. The following main tasks of factor analysis were formulated in accordance with variables ($V_1 – V_{10}$):

1) to investigate the structure of interconnections between input variables (each grouping of variables is determined by the factor which have the maximum value);
2) to identify factors that are the causes of the relationship between input variables;
3) to calculate the factor values as new, integral variables.

The factor analysis was conducted in the following sequence of steps:

1) the correlation matrix for all variables was calculated;
2) the principal factors were selected using the extraction method (principal component analysis);
3) the simplified factors' structure was identified using a rotation method (Varimax with Kaiser Normalization);
4) newly obtained factors were interpreted as integral variables.

The numerical value obtained (0.846) of the Kaiser-Meyer-Olkin sampling adequacy demonstrates a high sample correlation for the factor analysis. The Bartlett spherical criterion indicated a statistically significant result, since correlations between variables differed significantly from zero (Table 7). Table 8 lists the names of variables and grouping results (community).

Table 7. Measure of sampling adequacy and Bartlett's criterion

Kaiser-Meyer-Olkin measure of sampling adequacy		0.846
Bartlett's test of sphericity	Approx. Chi-Square	928.189
	df	0.045
	Sig	0.0001

Table 8. Variables and grouping results (community)

№	Names of variables	Input	Output
V_1	Teacher motivates students to study a theorem	1.000	0.649
V_2	A variety of techniques used by the teacher to motivate students to study theorems	1.000	0.693
V_3	Allocation of additional instruction time to practice the wording of the theorem	1.000	0.799
V_4	Practicing the wording of the theorem: identifying the explanatory part, hypothesis and conclusion of the theorem	1.000	0.844
V_5	Defining the form of the theorem statement (categorical, implicational)	1.000	0.793
V_6	Formulating the converse of a theorem	1.000	0.810
V_7	Formulating the opposite statement	1.000	0.799
V_8	Formulating the converse of a opposite statement	1.000	0.880
V_9	A variety of techniques used by the teacher in teaching the wording of the theorem	1.000	0.817
V_{10}	The number of difficulties students experience when working with the wording of the theorem	1.000	0.744

Table 9 shows the characteristics of the separate factors: the number, the sum of the squared loading, the percentage of the joint dispersion, which is caused by the factor, the corresponding cumulative percentage before and after loading.

Figure 2 shows an Eigenvalue graph that illustrates the three selected factors before loading.

Table 9. Total Variance Explained

Component	Initial Eigenvalues			Sums of Squared Loadings			Rotation Sums of Squared Loadings		
	Total	% of Variance	Cumulative %	Total	% of Variance	Cumulative %	Total	% of Variance	Cumulative %
1	5.202	52.023	52.023	5.202	52.023	52,023	4.511	45.114	45.114
2	1.583	15.828	67.851	1.583	15.828	67.851	1.714	17.142	62.256
3	1.042	10.421	78.273	1.042	10.421	78.273	1.602	16.016	78.273
4	0.670	6.704	84.977						
5	0.539	5.391	90.368						
6	0.350	3.502	93.870						
7	0.200	2.003	95.873						
8	0.160	1.597	97.470						
9	0.130	1.296	98.766						
10	0.123	1.234	100.00						

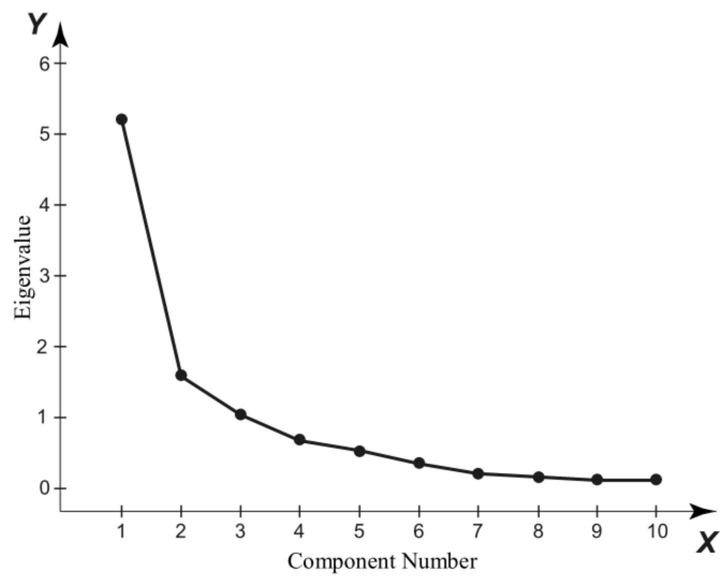

Figure 2. Initial Eigenvalues

Table 10. Rotated Component Matrix

№	Names of variables	Components		
		1	2	3
V_1	Teacher motivates students to study a theorem	0.736		
V_2	A variety of techniques used by the teacher to motivate students to study theorems		0,759	
V_3	Allocation of additional instruction time to practice the wording of the theorem	0.580	-0.583	
V_4	Practicing the wording of the theorem: identifying the explanatory part, hypothesis and conclusion of the theorem	0.613		0.630
V_5	Defining the form of the theorem statement (categorical, implicational)	0.807		
V_6	Formulating the converse of a theorem	0.892		
V_7	Formulating the opposite statement	0.873		
V_8	Formulating the converse of a opposite statement	0.907		
V_9	A variety of techniques used by the teacher in working with the wording of the theorem			-0.894
V_{10}	The number of difficulties students experience when working with the wording of the theorem		0.764	

Table 10 shows the factor loadings matrix after loading.

4. Discussion

The research allowed us to distinguish the three factors that influence the cooperation between teachers and students at the stage when the teacher motivates students to study a theorem and organizes their working with a wording of the theorem.

Factor 1 combines the following variables: teacher motivates students to study a theorem (V_1), allocation of additional instruction time to practice the wording of the theorem (V_3), practicing the wording of the theorem by breaking it into units (defining the explanatory part, hypothesis and conclusion of the theorem, a short record of the wording of the theorem, drawing a figure (V_4)); defining the form of the theorem statement (categorical, implicational (V_5)); formulating the converse of the theorem (V_6); formulating the opposite statement (V_7); formulating the converse of the opposite statement (V_8). This factor was named as the *realization of the invariant core of the traditional methodological scheme for working with the wording of the theorem.* This factor is the most influential; it involves the largest number of variables characterizing the stages in the traditional methodological scheme used at the stage of working with the wording of the theorem. Note that any 'variance' in the traditional methodological procedure used to work with the wording of the theorem in factor 1 is not reflected, so we use the term 'invariant core'. The variability in the implementation of its stages was reflected in factors 2 and 3.

Factor 2 combines variables: a variety of techniques used by the teacher to motivate students to study theorems (V_2); allocation of additional instruction time to practice the wording of the theorem (V_3); the number of difficulties that students faced in this work (V_{10}). Summarizing all variables of a factor, we assign it the name: *motivational resultant polymorphism in the traditional methodological scheme of work with the wording of the theorem.* Factor 2 describes variables that characterize discrete variance of a certain characteristic (polymorphism) in the traditional methodological procedure used to work with the wording of the theorem, especially those relating to the stage of motivation, the actual work with the wording of the theorem (the time characteristic of this phase) and the reflection stage (the number of difficulties that students face when they are learning how to deal with the wording of the theorem and to formulate the theorem). This explains its name.

Factor 3 combines the following variables: practicing the wording of the theorem following the traditional stages (identifying the explanatory part, the hypothesis and the conclusion of the theorem, the short entry of the theorem, including drawing the figure (V_4)); a variety of techniques used by the teacher in working with the wording of the theorem (V_9). This factor was named as: *bipolarity of collaboration between teachers and students when working on the wording of the theorem.* It reflects its multivector nature and what is more important, the bipolarity of the possible management and cooperation between the teacher and students while teaching and learning the wording of the theorem: from the complete and indisputable observance of the traditional sequence of stages (the allocation of the explanatory part, the hypothesis and the conclusion of the theorem, the short entry of the theorem, including drawing the figure) to disorder, didactic chaos in the selection of techniques that diversify such work.

Factors 2 and 3 to some extent are not obvious, reflecting hidden links between variables, but their analysis allows for some conclusions to be drawn. Teachers who used a clearly thought but limited number of techniques to teach the wording of the theorem had not less success than those who used many techniques that were incoherently ordered. Thus overly complicated approach caused unexpected difficulties for students, and as a result decreased student motivation.

5. Conclusions

The research findings obtained prove that teachers value the importance attached to teaching and learning theorems and their proofs and their role in developing students' general cognitive skills and logical thinking. However, present working teachers neglect the importance of teaching and learning theorems and their proofs for raising students' methodological, global, and cultural awareness.

Although, teachers are aware of the importance of the motivation stage for teaching and learning a theorem, they tend to neglect this stage or seem to motivate students non-systematically. The inadequate motivation for students to study theorems is becoming the first root barrier that has a negative impact to students' acquiring the appropriate mathematical and cognitive skills. Teachers explain that they lack enough instructional time to teach mathematics in general and to teach theorems and their proofs in particular.

We would like strongly state that the overwhelming majority of teachers realize how important it is to motivate students by emphasizing the practical significance of theorems. By showing students that they can apply theorems to solve real life problems, teachers prove the applied nature of mathematics. On the other hand, at this point students are not involved in conducting independent studies. Students lack the opportunity to learn how to organize their own research, to implement constructs, measurements, to generalize observations, to make assumptions about the properties of geometric figures for further proof or refutation.

The research proved that teachers have a positive attitude to practicing the wording of the theorem which is considered to be an important, didactically significant stage in teaching and learning theorems and their proofs. The data obtained indicate the existence of dependence between the time that teachers allocate to teaching the wording of the theorem and the number of difficulties that students face while learning theorems and their proofs. The study also proves that the increase in instruction time spent by teachers on teaching and learning the wording of the theorem does not guarantee that students will not have any difficulty in further understanding the content and the essence of the theorem.

The important factors that weigh the time efficiency when the teacher motivates learning a theorem and teaching its wording include:

1) teachers' passion to cover only the invariant core in the traditional methodological sequence of techniques at these stages of teaching a theorem;
2) motivational and resultant polymorphism in the traditional methodological sequence of techniques at this stages of teaching a theorem;
3) bipolarity of collaboration between teachers and students when working on the wording of the theorem: from the complete and indisputable observance of the traditional sequence of stages (the allocation of the explanatory part, the hypothesis, the conclusion of the theorem, the short entry of the theorem, including drawing the figure) to disorder, didactic chaos in the selection of techniques that diversify such work.

It was shown that only half of respondents competently motivate students to learn a theorem and to work with the wording of the theorem. This leads to the situation when the vast majority of students are lack the skill to identify all the data explicitly and implicitly presented in the wording of the theorem. There are the difficulties in constructing a graphic model, or to drawing a figure, or to presenting relevant arguments in the substantiation chains, since students previously had not identified the explanatory part, the hypothesis, or the conclusion of the theorem. At present the attention of the teachers is not focused at practices and techniques that allow educators to motivate students and didactically diversify the techniques that help to teach and to learn the wording of the theorem. None of the polled teachers offered their own practices at this stage, although some noted that techniques used at this stage depend on the level of the class academic and the complexity of the theorem.

We propose to improve teaching the theorem at the stage of motivating students to learn a theorem and also while working with the wording of the theorem in the following directions:

1) to develop students' positive attitude to working with the wording of the theorem;
2) to allocate extra instruction time to motivate students when teaching theorems in the school mathematics;
3) to help students understand that the direct and converse of an opposite statements are equivalent;
4) to develop students' mastery of the 'excavation' of the theorem mathematical content from its various semiotic layers (the complete text of the wording of the theorem, a shortened entry, including a graphic model and figure);
5) to find a balance between traditional and innovative approaches which will increase students' motivation to learn the theorems.

It is necessary to emphasize the importance of teaching and learning not only the wording of the theorem but also of its proof. It is considered to be a separate significant stage in studying theorem. Teachers are expected not only to demonstrate the proof of the theorem in class, but also motivate students to learn it. Initially, students have to be prepared to interpret the proof, which is in many cases skipped. Secondly, students should be motivated to master the proof since such work improves their own experience and cognitive skills. After all, learning to prove theorems develops student specific method of reasoning and the ability to use such reasoning in future life. The questions for the further research are: What are teachers' attitude, values and beliefs related to this stage of teaching the theorems? How do teachers incorporate this stage into their teaching?

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